MATH 504 HOMEWORK 4

Due Friday, October 12.

Problem 1. *Let κ be a regular uncountable cardinal.*

(1) *Give an example of an unbounded set in κ, that is not closed.*
(2) *Let $G : \kappa \to \kappa$ be any function. Show that the set $C = \{\gamma < \kappa \mid (\forall \alpha < \gamma)(G(\alpha) < \gamma)\}$ is closed.*

Problem 2. *Suppose that κ is an inaccessible cardinal. Show $L_\kappa \models GCH$. (Here you can use the Condensation Lemma as in the argument that $L \models GCH$.)*

Problem 3. *Let κ be a regular uncountable cardinal.*

(1) *Suppose that S is a stationary subset of κ and C is a club in κ. Show that $C \cap S$ is stationary. Give an example of a cardinal κ and two stationary subsets of κ that are disjoint.*
(2) *Suppose in addition that κ is inaccessible. Show that $\{\tau < \kappa \mid \tau \text{ is a cardinal }\}$ is club in κ.*

Problem 4. *Suppose that $\mathcal{F} \subset \mathcal{P}(\kappa)$ is a κ-complete normal ultrafilter on κ. I.e. $\mathcal{F}$ satisfies the following:*

- *for all $A \subset \kappa$, $A \in \mathcal{F}$ or $\kappa \setminus A \in \mathcal{F}$,*
- *if $\tau < \kappa$ and $\{A_\xi \mid \xi < \tau\}$ are sets in $\mathcal{F}$, then $\bigcap_{\xi<\tau} A_\xi \in \mathcal{F}$.*
- *for all regressive functions $f : \kappa \to \kappa$ (regressive means that $f(\alpha) < \alpha$ for all α), there is $\gamma < \kappa$ such that $f^{-1}(\gamma) := \{\alpha < \kappa \mid f(\alpha) = \gamma\}$ is in $\mathcal{F}$.*

Show that $\mathcal{F}$ is closed under diagonal intersections of length κ.

Remark: sometimes "normality" is defined as the property of being closed under diagonal intersections.